

Entropy in Spin Foam Models: The Statistical Calculation

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April 28, 2010

Abstract

Recently an idea for computing the entropy of black holes in the spin foam formalism has been introduced. Particularly complete calculations for the three dimensional euclidean BTZ black hole were done. The whole calculation is based on observables living at the horizon of the black hole universe. Departing from this idea of observables living at the horizon, we now go further and compute the entropy of BTZ black hole in the spirit of statistical mechanics. We compare both calculations and show that they are very interrelated and equally valid. This latter behaviour is certainly due to the importance of the observables.

1 Introduction

A way to compute the entropy of a black hole in the spin foam model description of quantum gravity has been introduced [1], [2]. In particular complete calculations were done for the case of the three dimensional euclidean BTZ black hole [2]. It is also important to point out to approaches that go on the same spirit. They compute the geometrical entropy associated to microstates of spin networks; in [3], the black hole entropy is studied in terms of quantum gravity and quantum information, that is in terms of entanglement of states, and in [4], [5] the entropy of a black hole is studied in terms of quantum surface states.

Here in the same context of our previous studies and as a continuation of our research program we now compute the entropy of the BTZ black hole [7] in the language of statistical mechanics. Departing from the observables idea

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which was developed in our previous work we construct the statistical partition function of our black hole universe(system).

We should of course compare both calculations and hope they give the same or at least equivalent results. We indeed show that they give the same results up to a constant factor. This leads us to conclude that we may be in the correct direction when thinking of a way to study entropy in the spin foam models.

We divide this paper as follows. In section 2 we briefly review the calculation of the entropy of euclidean BTZ black hole in the program of spin foam models of quantum gravity.

In section 3 we continue with our program for deriving a satisfactory way for computing the entropy of black holes in the spin foam models of quantum gravity. We derive a computation in the spirit of statistical mechanics. Moreover, our idea is based on the observables which live at the horizon and which were part of our calculation in previous work.

Finally in section 4 we conclude with a discussion of our paper and other related ideas.

2 BTZ black hole entropy and spin foams

We briefly review our idea for calculating the entropy of the euclidean BTZ black hole in the context of spin foams. For a nice introduction to spin foam models based on quantum groups see [6].

The euclidean BTZ black hole has an hyperbolic type metric which after some identifications it is topologically a solid torus with its horizon at the core [7]. For a deep study of the BTZ black hole see [8].

In the three dimensional case we have that the entropy of the BTZ black hole is given by

$$S \sim 2L \tag{1}$$

see [8]. After restoring factors \hbar and G it can be rewritten as

$$S \sim \frac{L}{4\hbar G} \tag{2}$$

The horizon is the core of the torus. We start by triangulating the Euclidean black hole. We consider triangulations of the solid torus containing interior edges, as we want the core of the torus(horizon) be formed by edges.

The idea is based on the three dimensional topological spin foam models where the irreducible representations of the quantum group $SU_q(2)$ play a role. They are given by the finite set $\{0, \frac{1}{2}, 1, \frac{3}{2} \dots \frac{r-2}{2}\}$ where $r \geq 3$.

A spin foam partition function of any triangulated three dimensional space-time M is given by

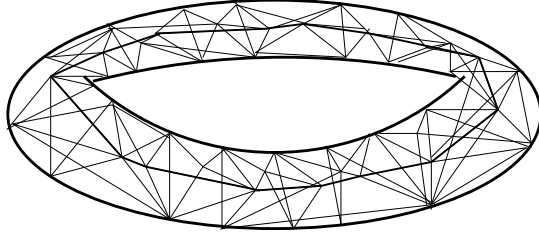


Figure 1: Triangulation of the BTZ Euclidean black hole together with its horizon

$$Z(M) = \sum_S \prod_{\text{edges}} \dim_q(j) \prod_{\text{tetrahedra}} \{6j\} \quad (3)$$

where the sum is carried over the set of all admissible states S and the amplitude $\{6j\}$ is the $6j$ symbol associated to the six labels of each tetrahedron. The quantum dimension is an amplitude associated to the edges.

The above partition function clearly can be applied to the BTZ black hole space-time $M = T^2$. But now consider the horizon of this black hole. We now think of the horizon as an observable \mathcal{O} in the following sense. Given the triangulation we consider the spin foam partition function with the difference that now we do not sum over the spins which label the horizon.

$$Z(T^2, \mathcal{O}(j_1, j_2, \dots, j_k)) = \sum_{S|\mathcal{O}} \prod_{\text{edges}} \dim_q(j) \prod_{\text{tetrahedra}} \{6j\} \quad (4)$$

The spin foam approach is analogous to a Feynman path integral. In this case it is the Feynman path integral of space-time. We therefore define the expectation value or correlation function as

$$W(T^2, \mathcal{O}) = \frac{Z(T^2, \mathcal{O})}{Z(T^2)} \quad (5)$$

This is therefore a function of the labels of the horizon. It is shown in [2] that for a particular triangulation of BTZ we have

$$W(T^2, \mathcal{O}) = \prod_m \frac{N_{i_m, j_m, \widehat{j_m}}}{\dim_q(\widehat{j_m})} \dim_q(i_m) \dim_q(j_m) \quad (6)$$

where $i_1, j_1, \dots, i_n, j_n$ are spins labelling the horizon, and $\widehat{j}_1, \dots, \widehat{j}_n$ are spins which label edges that do not belong to the horizon, however each triple $\{i_m, j_m, \widehat{j_m}\}$ forms a triangle. Our particular triangulation has a horizon with an even number of edges.

We propose that the entropy is given by the logarithm of formula (5)

$$S = \sum_{m=1}^{2n} \log(\dim_q(j_m)) - \sum_{k=1}^n \log(\dim_q(\widehat{j_k})) \quad (7)$$

It can be seen that the main contribution is given when the spins $\widehat{j_m}$ are zero. This implies that each pair of the edges of the horizon are equal $i_1 = j_1, i_2 = j_2, \dots, i_n = j_n$. The labels of the edges of the horizon by spins j are interpreted in the spin foam model as giving a discrete length given by $j + \frac{1}{2}$.

The horizon is discrete and formed by edges with spins j_1, \dots, j_{2n} .¹

We have a constraint, since we want the sum of all the discrete lengths of the horizon be L

$$\left(j_1 + \frac{1}{2}\right) + \left(j_2 + \frac{1}{2}\right) \cdots + \left(j_{2n} + \frac{1}{2}\right) = L \quad (8)$$

Observe that particularly when all of the spins at the horizon are equal we have

$$2nj + \frac{2n}{2} = L \quad (9)$$

which implies that

$$n = \frac{L}{(2j+1)} \quad (10)$$

Consider the case in which the number of spins we have goes to infinity, that is, when we go from the quantum group $SU_q(2)$ to the classical one $SU(2)$, such that $\dim_q(j) \rightarrow \dim(j) = (2j+1)$

The entropy is then given by

$$S \sim L \frac{2 \log(2j+1)}{(2j+1)} \quad (11)$$

Finally it can be seen that the main contribution is given when $j = 1$, that is,

$$S \sim 2L \frac{\log(3)}{3} \quad (12)$$

which is proportional to the length of the horizon and to equation (1).

The spin foam model procedure described in this section uses the quantum group $SU_q(2)$; at the end we took the limit $q \rightarrow 1$. Our calculation can be thought as having a regularisation procedure; it could be interesting to think if a different regularisation procedure could be used as an alternative method to derive the entropy of the black hole in a similar spirit as ours.

¹We have just relabel edges $i_1, j_1, \dots, i_n, j_n$ by j_1, j_2, \dots, j_{2n}

3 Entropy in terms of the statistical partition function

In this section, following our previous idea of the calculation of the entropy in the spin foam models of quantum gravity, we continue our proposal by deriving the entropy in a statistical spirit. In other words, we now consider the statistical partition function of our model.

The horizon is an observable in our picture. This means that the microstates live at the horizon. And we saw that when considering the main contribution to the entropy in the spin foam formalism we should only care about what happens at the horizon, that is, it only matters the spins which label them.

Since our approach is based on the spin foam models with cosmological constant² we only have a finite number of half integer spins $\{0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \frac{r-2}{2}\}$ where $r \geq 3$.

These spins are interpreted as giving a discrete length to any labelled edge which as we saw contribute to the entropy of the black hole. Recall that a spin j labelling an edge is interpreted as having a discrete length $j + \frac{1}{2}$. For convenience we consider the length written as $\frac{2j+1}{2}$, and call the integer $l_j = 2j + 1$.

Our horizon is labelled, and must have length L . Therefore formula (8) of section 3 must be satisfied. We rewrite it here and therefore have that

$$n_1 \frac{l_0}{2} + n_2 \frac{l_{1/2}}{2} + \dots + n_{r-1} \frac{l_{(r-2)/2}}{2} = L \quad (13)$$

where clearly $l_0 = 1, l_{1/2} = 2, \dots, l_{(r-2)/2} = (r-1)$ and n_1, n_2, \dots, n_{r-1} are the number of edges labelled with spins $0, 1/2, \dots, (r-2)/2$ respectively. Interpret formula (13) as in the case of the harmonic oscillator where now we have energy given by length and occupation numbers given by the finite set of $\{l_j\}$.

We can consider a Boltzman parameter β in which the half term in the sum (13) is absorbed, that is, $\beta = \frac{1}{2kT}$. We think of our model as an isolated system, similar to a gas made of an arbitrary number of photons which obeys a Bose-Einstein statistics. In our case the analogy is to consider our horizon to be formed by an arbitrary number of edges and each edge can be in any length state, that is, we can have any number of edges labelled with spin $1/2$, some others labelled with spin 1 , and so on.

Therefore the statistical partition function of our model is given by

²Positive cosmological constant in this case although the BTZ black hole has negative cosmological constant, we are not concern with this issue here, but the idea is what matters for us at the moment

$$\begin{aligned}
Z &= \sum_{n_1, n_2, \dots, n_{r-1}} \exp[-\beta(l_0 n_1 + l_{1/2} n_2 + \dots l_{(r-2)/2} n_{r-1})] \\
&= \sum_{n_1, n_2, \dots, n_{r-1}} \exp[-\beta(n_1 + 2n_2 + \dots (r-1)n_{r-1})]
\end{aligned} \tag{14}$$

We want to consider the case of a black hole with very large length, that is, when the length goes to infinity. Recall that our set of spins is finite, therefore we need to consider a horizon with a very large number of edges which implies that n_1, n_2, \dots, n_{r-1} are unrestricted and can go to infinity.

The partition sum can be rewritten

$$\begin{aligned}
Z &= \sum_{n_1=0}^{\infty} \exp[-\beta(n_1)] \sum_{n_2=0}^{\infty} \exp[-\beta(2n_2)] \dots \sum_{n_{r-1}=0}^{\infty} \exp[-\beta((r-1)n_{r-1})] \\
&= \prod_{m=1}^{r-1} \sum_{n_m=0}^{\infty} \exp[-\beta(mn_m)] = \prod_{m=1}^{r-1} \sum_{n_m=0}^{\infty} \exp(-\beta)^{mn_m}
\end{aligned} \tag{15}$$

If we consider the complex variable $z = \exp(-\beta)$ in such a way that $|z| < 1$ we have that

$$\sum_{n_m=0}^{\infty} \exp(-\beta)^{mn_m} = \frac{1}{1 - e^{-\beta m}} \tag{16}$$

and the partition function is now given by

$$Z = \prod_{m=1}^{r-1} \frac{1}{1 - e^{-\beta m}} \tag{17}$$

Before we continue let us very briefly make the following mathematical observation.

The partition function written as in formula (17) is a generating function of the number of partitions of the positive integers \mathbf{Z}^+ in terms of the finite set $\{l_0 = 1, l_{1/2} = 2, \dots, l_{(r-2)/2} = (r-1)\}$. We already know that this finite set is related to the set of the discrete lengths associated to the spins from which we are labelling the edges of our horizon.³ This gives a very nice connection to analytic number theory [9]. We will come back to this relation later on in the following section.

We can also state that analogous to the gas made of photons the Planck distribution can be derived from formula (16) and it is given by

³ $j + \frac{1}{2} = l_j/2$

$$\bar{n}_m = \frac{1}{e^{\beta m} - 1} \quad (18)$$

which gives information about the statistical equilibrium. That is, the most probable partition corresponds to these numbers, given by the lowest spins of the finite set $\{l_0 = 1, l_{1/2} = 2, \dots, l_{(r-2)/2} = (r-1)\}$.

The free energy is given by $F = -kT \ln Z$. Substituting $\beta = \frac{1}{2kT}$ we have

$$F = -kT \ln \left[\prod_{m=1}^{r-1} \frac{1}{1 - e^{-\frac{m}{2kT}}} \right] = -kT \sum_{m=1}^{r-1} \ln \left(\frac{1}{1 - e^{-\frac{m}{2kT}}} \right) \quad (19)$$

which gives

$$F = kT \sum_{m=1}^{r-1} \ln(1 - e^{-\frac{m}{2kT}}) \quad (20)$$

The entropy is then given by

$$\begin{aligned} S &= -\frac{\partial F}{\partial T} = -\left[k \sum_{m=1}^{r-1} \ln(1 - e^{-\frac{m}{2kT}}) - kT \sum_{m=1}^{r-1} \frac{\frac{m}{2kT^2} e^{-\frac{m}{2kT}}}{(1 - e^{-\frac{m}{2kT}})} \right] \\ &= \frac{1}{2T} \sum_{m=1}^{r-1} \frac{m e^{-\frac{m}{2kT}}}{(1 - e^{-\frac{m}{2kT}})} - k \sum_{m=1}^{r-1} \ln(1 - e^{-\frac{m}{2kT}}) \end{aligned} \quad (21)$$

Now, technically the set of spins (equivalently the set of $\{l_j\}$) we have depend on the quantum group we choose $SU_q(2)$. We can choose to take the limit when $q \rightarrow 1$, that is when going to the classical group $SU(2)$; just as we did at the end of our calculation in section 2.

Then the entropy can be considered to be given by

$$S = \frac{1}{2T} \sum_{m=1}^{\infty} \frac{m e^{-\frac{m}{2kT}}}{(1 - e^{-\frac{m}{2kT}})} - k \sum_{m=1}^{\infty} \ln(1 - e^{-\frac{m}{2kT}}) \quad (22)$$

If we go further and approximate the sums by integrals we have

$$S = \frac{1}{2T} \int_1^{\infty} \frac{m e^{-\frac{m}{2kT}} dm}{(1 - e^{-\frac{m}{2kT}})} - k \int_1^{\infty} \ln(1 - e^{-\frac{m}{2kT}}) dm \quad (23)$$

The integrals can be computed for example with a mathematical program and then obtain that the entropy is given by

$$\begin{aligned} S &= \frac{1}{2T} \left(\frac{1}{2} + \frac{4\pi^2 k^2 T^2}{3} - 2kT \ln(1 - e^{\frac{1}{2kT}}) - 4k^2 T^2 \text{Li}_2(e^{\frac{1}{2kT}}) \right) \\ &\quad - k \left(-\frac{1}{4kT} - \frac{2\pi^2 kT}{3} + \ln(1 - e^{\frac{1}{2kT}}) - \ln(1 - e^{\frac{-1}{2kT}}) + 2kT \text{Li}_2(e^{\frac{1}{2kT}}) \right) \end{aligned} \quad (24)$$

where the terms $\text{Li}_2(e^{\frac{1}{2kT}})$ are called polylogarithmic integrals given by

$$\text{Li}_2(e^{\frac{1}{2kT}}) = \int_0^{e^{\frac{1}{2kT}}} \frac{\ln(1-x)}{x} dx \quad (25)$$

Arranging terms, the entropy is given by

$$S = \frac{4\pi^2 k^2 T}{6} + \frac{1}{2T} - 2k \ln(1 - e^{\frac{1}{2kT}}) + k \ln(1 - e^{\frac{-1}{2kT}}) - 4k^2 T \text{Li}_2(e^{\frac{1}{2kT}}) \quad (26)$$

Our spin foam model description of black hole entropy is a first step towards the microscopic description in terms of this quantum gravity direction. We are also considering an Euclidean black hole and our model is in fact a toy model. Let us go a bit further and suppose for a moment that the relation between temperature and length is fulfilled. For the BTZ black hole the temperature is given by $T = \frac{R}{2\pi}$, where R is the radius of the horizon, see for instance [8]. In the present case, our statistical partition function (14), can be thought as a mathematical partition of the number $2L$, which therefore in this case leads us to consider the temperature to be given by $T = \frac{R}{\pi}$. With this value of the temperature the entropy (first term) is therefore given by

$$S \sim \frac{4\pi r}{6} k^2 = 2L \frac{k^2}{6} \quad (27)$$

which is indeed proportional to the formula obtained in our previous work [2], given here by formula (12). They will be exactly equal if we have units in which we consider $k^2 = 2 \log(3)$. Moreover, after restoring the constants \hbar and G it can be argued that we are indeed obtaining the correct entropy of $\frac{L}{4\hbar G}$ up to a constant factor.

4 Discussion

We have calculated the entropy of the euclidean three dimensional black hole in the spin foam formalism [2]. We reviewed it in section 2 and extended the idea to the spin foam statistical calculation based on the observables. The observables played a significant role. The interpretation is that it only matters what happens at the horizon, when considering the entropy.

Let us mention some interesting ideas that come up here. How can all our calculations be extended to an isolated four dimensional black hole universe? That is, develop a spin foam approach to entropy of four dimensional black holes [1], as it is done in Loop Quantum Gravity [10], [11], [12], [13], [14], [15].

We may have naively chosen units such that $k^2 = 2 \log(3)$ so that our formulae (12) and (26) match. What does it have to do with the fact that the spectrum of a quantum non rotating black hole such as Schwarzschild is evenly spaced containing a factor $\log(3)$ as discussed in [16], and later in [17].

Another interesting fact is the relation of counting microstates which account for a fixed area and partitions of integers in the field of number theory, principally analytic [9]. For instance recent developments about these number theory relations for the case of Loop Quantum Gravity were introduced by [15] and continued in [18], [19].

Now particularly for this paper the relation is as follows. Suppose we want to count the number of undistinguishable microstates which account for a fixed length horizon L . Then we would have to proceed as follows.

The statistical partition function written as formula (17), is the generating function for the number of partitions of the positive integers in terms of the finite set $\{l_0 = 1, l_{1/2} = 2, \dots, l_{(r-2)/2} = (r-1)\}$.

Such formula can be expanded as follows

$$Z = \prod_{m=1}^{r-1} \frac{1}{1 - e^{-\beta m}} = \sum_{n=0}^{\infty} p_{r-1}(n) e^{-\beta n} \quad (28)$$

where p_{r-1} is the number of partitions of the number n in terms of integers not exceeding $r-1$, that is in terms of our finite set $\{l_0 = 1, l_{1/2} = 2, \dots, l_{(r-2)/2} = (r-1)\}$.

If considering a large horizon length L , then counting the number of partitions of such number L gives us back $p_{r-1}(L)$. What we would be really counting with $p_{r-1}(L)$ is the number of undistinguishable microstates of the black hole. This means that we would not be distinguishing between a given microstate which accounts for a fixed length and a permutation of this microstate.

In this case the number of microstates goes asymptotically as follows

$$p_{r-1}(L) = \frac{L^{r-2}}{(r-1)!(r-2)!} \quad (29)$$

as can be seen in [9]. Therefore if we want to calculate the entropy when the microstates are thought as undistinguishable, it is just mainly given by the logarithm of formula (28).

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